Statistics of the cumulative phase of microwave radiation in random media

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We determine the cumulative phase of microwave radiation transmitted through a sample of randomly positioned polystyrene spheres in measurements of the field versus frequency and investigate its statistics. Its probability distribution is Gaussian at all frequencies with a variance which is nearly equal to the ensemble average of the phase. These results are consistent with our observation that the correlation function of the phase derivative with the ensemble average value of the phase is nearly the same exponential function for diffusing waves over a wide frequency range. Finally, we indicate ways in which the study of the cumulative phase can elucidate wave transport in random systems. [S1063-651X(97)12508-9]

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I. INTRODUCTION

Large fluctuations in optical and electronic properties of statistically equivalent realizations of a random medium are a consequence of interference and reflect the essential role played by the phase in mesoscopic physics [1–4]. Recently the fascinating topological structure of phase maps has been investigated [5–7], but the statistics of the total phase φ at a point and its frequency dependence in an ensemble of random systems has not been studied. This may be because the distribution of the phase modulo 2π in the field $E\ \exp(i\varphi_{2\pi})$, which is the phase that is ordinarily measured, is flat in the interval $[-\pi,+\pi]$ for diffusive waves. This is seen in the probability distribution shown in Fig. 1 of $\varphi_{2\pi}$ for microwave radiation transmitted through the random polystyrene sample described below. Another reason the phase has not been extensively investigated is that the ensemble average of the field decays rapidly as the sample thickness is increased. As a result, studies of wave propagation in random media have focused on the magnitude of the field E or on its square, the intensity, rather than on the phase. Here we show that the total phase accumulated by the wave in traversing the medium [8] is a rich statistical quantity which is central to the understanding of wave transport in random systems. We present measurements of the frequency dependence of the average phase, its probability distribution, and the correlation function of the phase derivative for microwave radiation transmitted through random media. These studies allow us to explain the relationship of the ensemble average value of the phase to its variance. We then outline some of the ways in which φ reflects wave dynamics and the density of states in random systems.

II. EXPERIMENTAL PROCEDURE

The phase of microwave radiation transmitted through a sample of randomly positioned 1/2-inch polystyrene spheres at a volume filling fraction of 0.52 is measured using a Hewlett-Packard 8722C network analyzer as it is swept from 3 to 26 GHz. The sample of length $L=110$ cm is contained within a 7.6-cm-diam copper tube. The incident wave is emitted using a broadband horn peaked at 18 GHz and the transmitted field is picked up with a wire antenna at the output surface. Calibration of the instrument at the input sets the phase reference. Measurements of E and $\varphi_{2\pi}$, from 18.8 to 19.0 GHz for a single configuration are shown in Fig. 2. Data are taken at frequency intervals of 625 kHz. The phase modulo 2π is generally seen to increase with frequency in a piecewise fashion.

In a multiply scattering medium, the complex field at a given observation point may be expressed as the sum over all partial waves emanating from the source, $Ee^{i\varphi}=\Sigma p_\alpha e^{i\varphi_\alpha}$ where φ_α is the phase accumulated along the wave path α and p_α its magnitude. The cumulative phase of the resulting field can be expressed as $\varphi=\varphi_{2\pi}+2n\pi$, where n is an integer. This integer can be determined by following the phase rollup starting from low frequencies where the phase approaches zero. Measurements are made using frequency in-

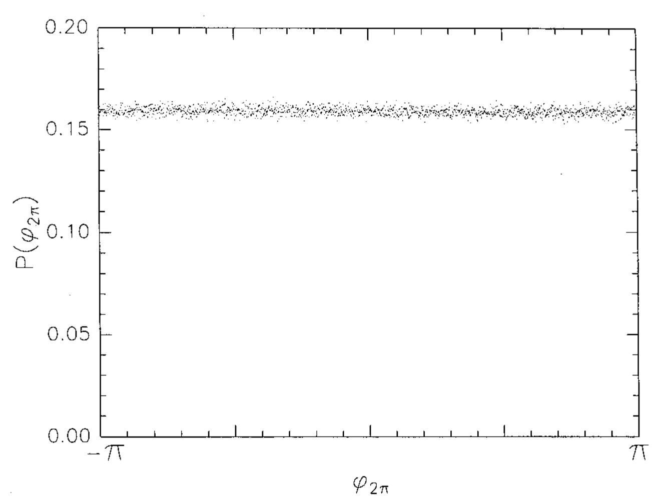

FIG. 1. Probability distribution of φ modulo 2π.

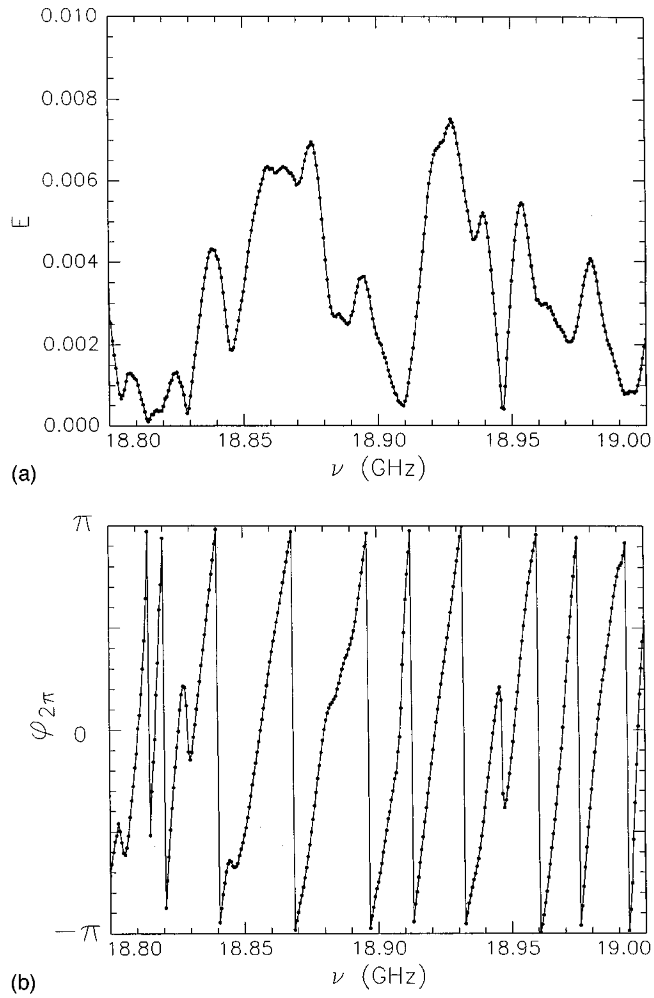

FIG. 2. Measurements of (a) the field magnitude and (b) its phase modulo 2π for a single sample configuration. The line through the data is present as a guide to the eye.

crements which are small enough that the change in the measured phase $\varphi_{2\pi}$ is generally much less than π rad. Occasional large jumps in $\varphi_{2\pi}$ of up to $\pm\pi$ rad are observed when phase singularities in the speckle pattern, associated with a zero of the field [5–7], sweep past the detector as the frequency increases. A jump in $\varphi_{2\pi}$, which is equal to $\pm\pi$ rad within the uncertainty of measurements, would give rise to an indeterminacy in the cumulative phase [6,7] but is never observed.

The zero of φ occurs at frequency $\nu=0$. This point is estimated by extrapolating the phase measured in the interval between 3 and 6 GHz to zero frequency. New sample configurations are created after each spectrum is taken by rotating the copper tube [9].

III. RESULTS AND DISCUSSION

Following the procedure outlined above, we construct the total phase accumulated for each of 581 configurations measured from 6 to 26 GHz and obtain the average over these configurations shown in Fig. 3(a). The fluctuations of the phase from its ensemble average value, $\delta\varphi=\varphi-\langle\varphi\rangle$, are shown in Fig. 3(b) for three configurations. The dependence

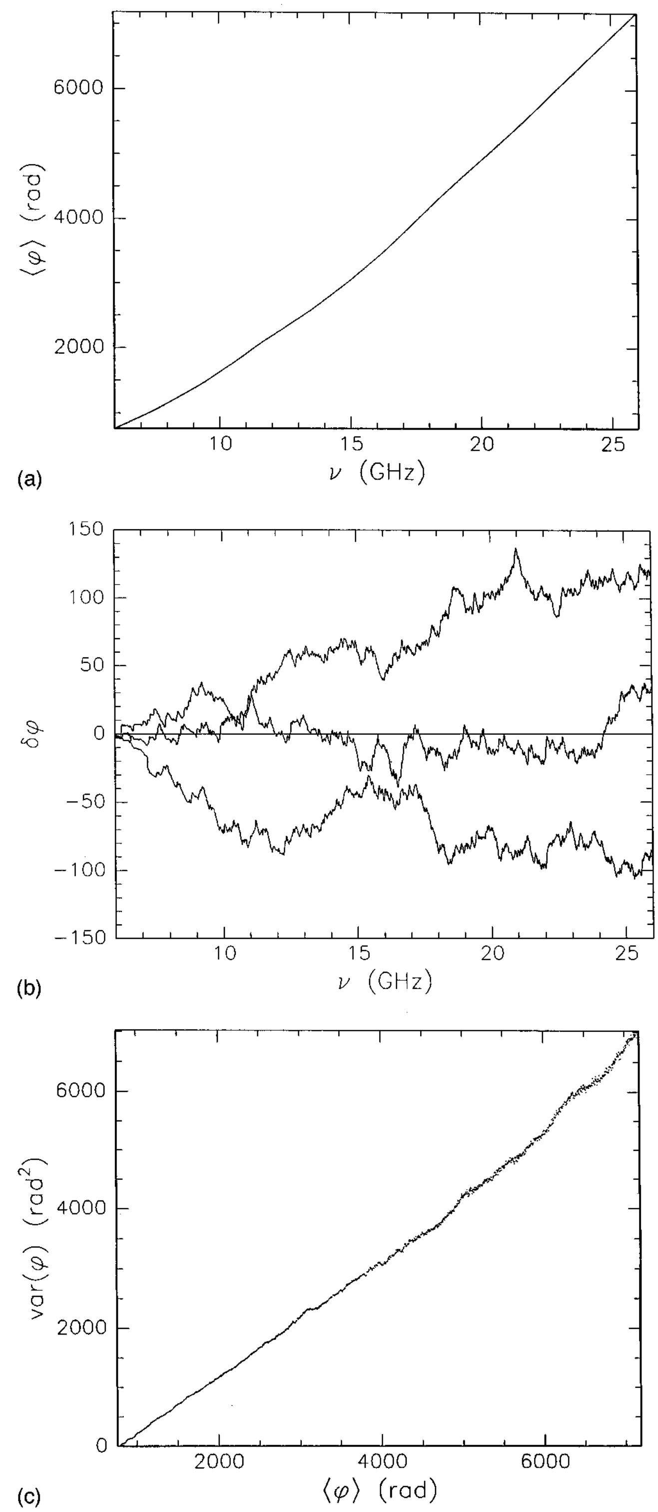

FIG. 3. (a) Variation of the ensemble average cumulative phase $\langle\varphi\rangle$ for an ensemble of 581 configurations from 6 to 26 GHz. (b) Fluctuations of the phase from its ensemble average value, $\delta\varphi=\varphi-\langle\varphi\rangle$, for three different configurations. (c) Dependence of the variance in cumulative phase upon the ensemble average cumulative phase.

of the variance of the phase $\mathrm{var}(\varphi)$ upon its average $\langle\varphi\rangle$ is shown in Fig. 3(c).

The field is a random variable representing the sum over a large number of partial waves with random phases. To ex-

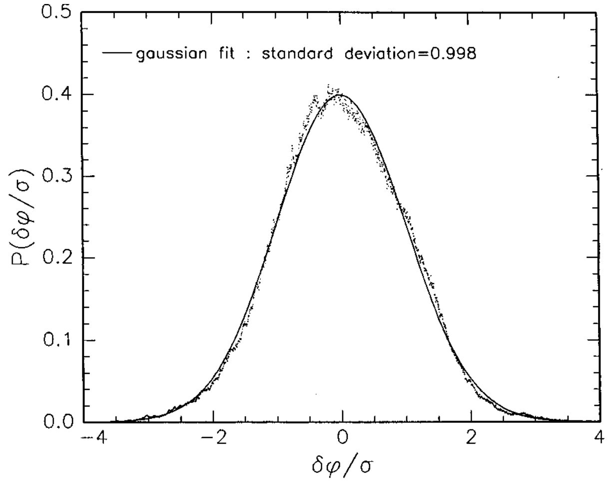

FIG. 4. Probability distribution of $\xi(\nu)=\delta\varphi(\nu)/\sigma(\nu)$ calculated at each frequency (from 6 to 26 GHz) for every configuration. The line through the data is a fit of a Gaussian to the data. This fit gives a standard deviation of 0.998.

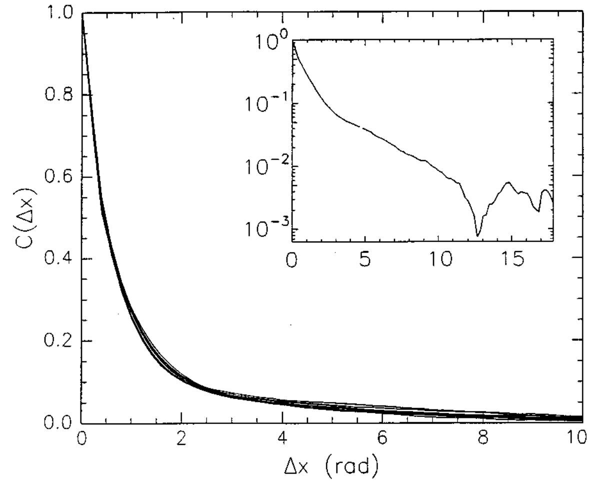

FIG. 5. Normalized cumulant correlation function $C(\Delta x)$ of the phase derivative with average phase x for 15 intervals corresponding to frequencies running from 11.47 to 25.15 GHz. A semilogarithmic plot of $C_{18}(\Delta x)$ is shown in the inset.

amine the character of the distribution of cumulative phases resulting from this phasor sum, we compute the probability distribution $P(\varphi)$, using the cumulative phase measured at each frequency (from 6 to 26 GHz) for every configuration. This probability distribution is presented in Fig. 4 on a single plot in terms of the variable $\xi(\nu)=\delta\varphi(\nu)/\sigma(\nu)$, where $\sigma=[\mathrm{var}(\varphi)]^{1/2}$. A Gaussian fit to the measured probability distribution $P(\xi)$ gives the solid line through the data shown in Fig. 4 with standard deviation 0.998.

In order to understand this result, we consider the cumulant correlation function of the phase derivative. The underlying character of fluctuations in the phase is revealed in a way that may be independent of frequency range and sample characteristics when we change variables from the frequency ν to $x=\langle\varphi(\nu)\rangle$, the average increment in the cumulative phase. We take the increment from the value of $\langle\varphi(\nu)\rangle$ at 6 GHz, which is the lowest value of the frequency in the field spectra measured in this study of fluctuations in the phase. This mapping is allowed because $\langle\varphi(\nu)\rangle$ is a monotonically increasing function of frequency. The new variable x runs from 0 to 6425 rad for the frequency range 6–26 GHz. This range of phase change is divided into 20 identical phase intervals. This is small enough for the statistical process to be stationary but large enough that the quality of the statistics is improved when we average over both sample configurations and over x within a given interval. The comparison of the cumulant correlation functions in different frequency ranges is facilitated by considering the normalized correlation function. Writing the phase derivative as $\varphi'=d\varphi/dx$ and noting that $\langle\varphi'\rangle=1$, we express the normalized cumulant correlation function of the phase derivative with x in the ith interval as follows:

$$C_i(\Delta x)=\langle\langle A(x)A(x+\Delta x)\rangle_i\rangle, \tag{1}$$

where

$$A(x)=[\varphi'(x)-1]/\sqrt{\langle[\varphi'(x)-1]^2\rangle}.$$

The symbol $\langle\ \rangle$ represents the average over configurations and $\langle\ \rangle_i$ the average over x within the ith interval. The result for 15 intervals (i from 5 to 19 corresponding to frequencies running from 11.47 to 25.15 GHz) is shown in Fig. 5. The inset in the figure gives a semilogarithmic plot of C for interval $i=18$. The plots of the correlation function overlap to a significant extent. At lower frequencies ($i=1$–4) the diffusive regime is not completely established and the corresponding plots of the correlation function do not overlap. Changes in the correlation function which develop at higher frequencies are the sources of the deviation from the linear relationship between the variance and average of the phase seen in Fig. 3(c). The half width of the correlation function is $\delta x=0.5$ rad. Because the degree of correlation of the phase derivative exhibits a rapid decay with an exponential tail, the cumulative phase at a given frequency is essentially a sum over a large number of statistically independent increments. The Gaussian distribution of $\delta\varphi$ shown in Fig. 4, obtained by sampling over configuration and over frequency, is thus a consequence of the central limit theorem.

When a correlation function $C(\Delta x)$ falls more rapidly than $(\Delta x)^{-1}$, the variance of the increment in x over some range is proportional to x over a frequency range in which the correlation function is stationary [10]. Only the constant of proportionality between the variance and x is modified by the correlation function. Thus, the nearly linear variation of var(φ) with $\langle\varphi\rangle$, seen in Fig. 3(c) for much of the frequency range investigated, is related to short range of the correlation function of the phase derivative with average phase shift and to the independence of the correlation function upon the frequency range [10].

The slope of the variation of var(φ) with $\langle\varphi\rangle$ is related to the range of the correlation function. This is seen by considering the variance of φ at a specific value of x, X:

$$\mathrm{var}(\varphi(X))=\langle|\varphi(X)-X|^2\rangle=\left\langle\left|\int_0^X dx[\varphi'(x)-1]\right|^2\right\rangle. \tag{2}$$

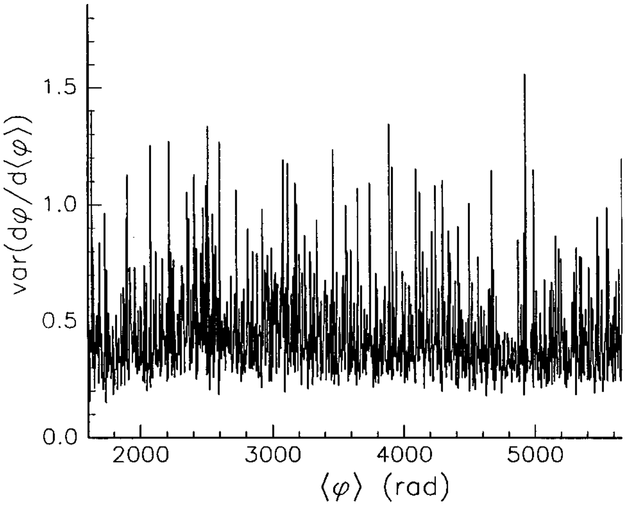

FIG. 6. The variance of the phase derivative vs average cumulative phase.

The RHS of Eq. (2) can be expressed in terms of the cumulant correlation function $\widetilde{C}(x,\Delta x)=\langle\langle[\phi'(x)-1][\phi'(x+\Delta x)-1\rangle_1\rangle$ of the phase derivative without the normalization factor used in Eq. (1):

$$\mathrm{var}(\varphi)=\int_{-X}^{0}d(\Delta x)\int_{-\Delta X}^{X}\widetilde{C}(x,\Delta x)dx+\int_{0}^{X}d(\Delta x)\int_{0}^{X-\Delta x}\widetilde{C}(x,\Delta x)dx. \tag{3}$$

The results in Eq. (3) can be put into a particularly simple form when $\widetilde{C}$ is independent of frequency range. Since we have already seen in Fig. 5 that C is independent of frequency range, $\widetilde{C}$ will be independent of frequency range as long as var(φ') is. In Fig. 6 we plot var(φ') for the 15 intervals for which data are shown in Fig. 5. The fluctuations in the figure are a result of the noise in computing the average for 581 configurations. We find that the average value of var(φ') is 0.46 in each of the frequency ranges. Therefore $C(x,\Delta x)$ and $\widetilde{C}(x,\Delta x)$ are proportional. As a result $\widetilde{C}$ is independent of x. We have then

$$\mathrm{var}[\varphi(X)]=\int_{-X}^{0}(X+\Delta x)\widetilde{C}(\Delta x)d(\Delta x)+\int_{0}^{X}(X-\Delta x)\widetilde{C}(\Delta x)d(\Delta x). \tag{4}$$

Since $\widetilde{C}(-\Delta x)=\widetilde{C}(\Delta x)$, we have

$$\mathrm{var}[\varphi(X)]=2X\int_{0}^{X}\widetilde{C}(\Delta x)d(\Delta x). \tag{5}$$

Because $\widetilde{C}$ falls off rapidly, the integral $\int_0^X\widetilde{C}(\Delta x)d(\Delta x)$ does not depend upon the upper limit and is a constant independent of X. Thus, var$[\varphi(X)]$ is proportional to X. The value of the integral in Eq. (5) of $\widetilde{C}(\Delta x)$ in any interval is obtained by multiplying the integral of the correlation function in Fig. 5 by the average value of var(φ') in that interval.

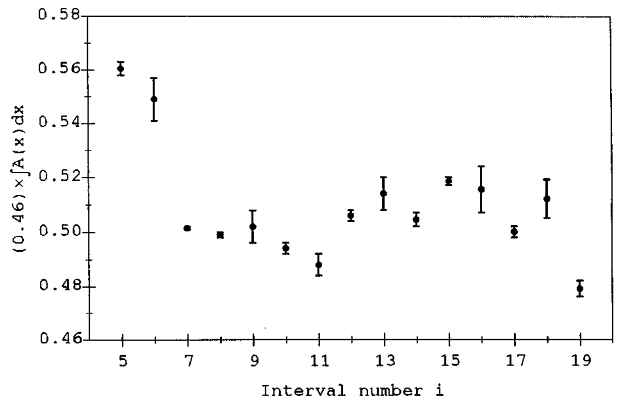

FIG. 7. The integral over the average phase of the correlation function of the phase derivative evaluated in terms of the integral of the product of the average variance of the phase derivative and the normalized correlation function. The error bars take into account the uncertainty in the integral of the correlation function in view of the noise as seen in the inset of Fig. 5.

The results for 15 intervals from $i=5-19$ GHz are plotted in Fig. 7. We find that the integral of the phase derivative correlation function is nearly constant with an average value over the frequency ranges considered of 0.50. As a result we find that, for our sample,

$$\mathrm{var}[\varphi(X)]=X, \tag{6}$$

in good agreement with Fig. 3(c). A linear fit of the results in Fig. 3(c) gives var(φ)$=1.01x$ for $x<5000$ with somewhat larger values for the prefactor for higher values of x. The discrepancy may be the result of contributions to the integral of the correlation function in Fig. 5 for values of Δx greater than those for which the correlation function was above the noise.

IV. CONCLUSION

The significance of the cumulative phase is seen by considering some of the ways it enters into a description of wave transport in random media. The derivative of the phase with angular frequency provides a convenient window on wave dynamics in random media. We expect that the dwell time of a narrow bandwidth pulse centered at the carrier frequency ω incident in one channel and emerging from another channel is equal to $d\varphi/d\omega$ at ω for the field in the outgoing channel. Thus the nearly linear increase of $\langle\varphi\rangle$ seen in Fig. 3(a) reflects a nearly constant average passage time for the wave. This can be understood by considering the dwell time in our strongly absorbing sample, the length of which is longer than the absorption length, $L>L_a$, where $La=\sqrt{D\tau_a}$, D is the diffusion constant and τ_a the absorption time [12]. In this case, the average transit time is proportional to the product of the transit time through one absorption length, L_a^2/D, and the number of absorption lengths in the sample, L/L_a. Thus, $\langle\tau\rangle\sim LL_a/D\sim L\sqrt{\tau_a/D}$. Measurements in this sample have shown that τ_a and D happened to be proportional over a broad frequency range [11]. Thus the linear behavior of $\langle\varphi\rangle$ with frequency is a consequence of the properties of this particular system.

The integrated energy within the sample due to an excitation of a particular incident channel is proportional to the dwell time for that channel. Thus, the density of states within the sample, which is proportional to the volume integral of the intensity within a medium in which all incident channels have equal energy, is proportional to the sum over all input channels of the dwell time of photons in each of these channels [12,13]. Measurements of the phase derivative thus can yield the density of states in a random medium.

In summary, we have measured the cumulative phase in a random medium and investigated its statistics. Its average increases monotonically with frequency and its variance is nearly equal to the average cumulative phase. We show that the Gaussian shape of its probability distribution at any frequency is a direct consequence of the unchanging form of the correlation function of the phase derivative in our sample. The cumulative phase is a key statistical parameter which, despite the complexity of the interference process, allows a statistical study of wave dynamics and can be used to determine the density of states in random media.

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